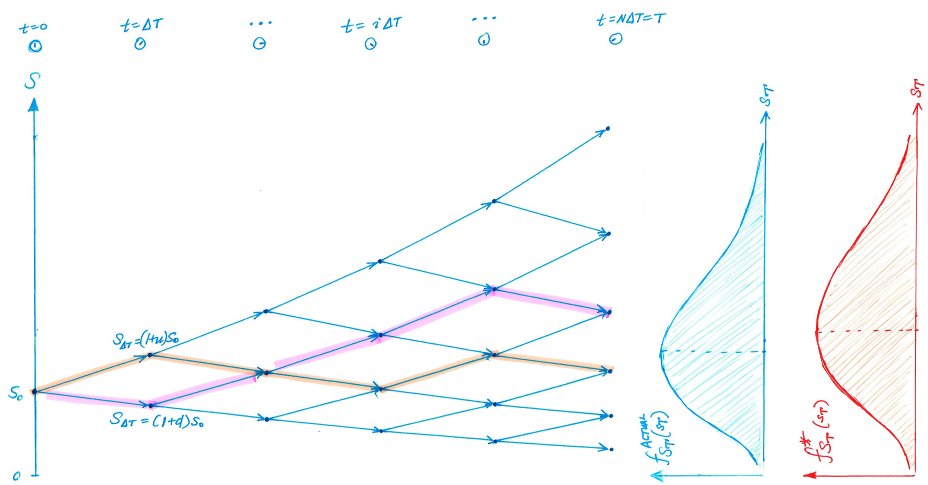
Continuous-time limit and Black-Scholes formula

Continuous-time limit



$$S_T|S_t = S_0 \cdot \frac{S_1}{S_0} \cdot \frac{S_2}{S_1} \cdots \frac{S_{n \text{ or "}T\text{"}}}{S_{n-1}}$$

The final asset value equals the initial asset value multiplied by a product of a bunch of independent factors of comparable size.

$$\log S_T\,|S_t = \log(S_0) + \log\left(\frac{S_1}{S_0}\right) + \log\left(\frac{S_2}{S_1}\right) + \cdots + \log\left(\frac{S_{n \text{ or "}T\text{"}}}{S_{n-1}}\right)$$

The log of the final asset value equals the log of the initial asset value plus a sum of a bunch of independent terms of comparable size.

Using central-limit theorem,

$$\log S_T\,|S_t \sim \mathcal{N}\left(\mu_{\log S_T|S_t}, \sigma^2_{\log S_T|S_t}\right)$$

FTAP using ZCB numeraire

$$\frac{S_t}{Z(t,T)} = \mathbb{E}_*\left[\frac{S_T}{Z(T,T)}\middle| S_t\right]$$

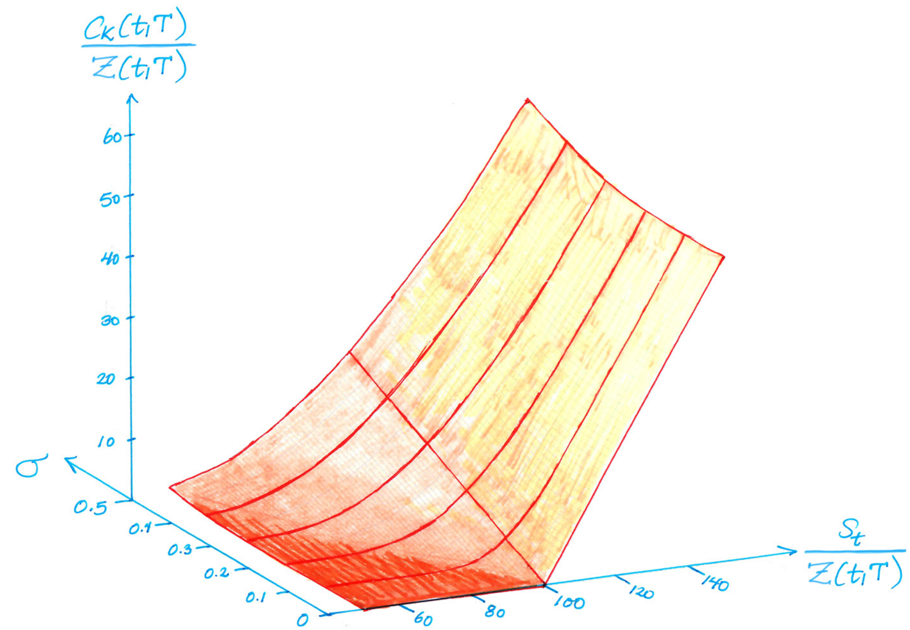
$$\log S_T\,|S_t \sim \mathcal{N}\left(\log S_t + \left(r - \frac{1}{2}\sigma^2\right)(T-t), \sigma^2(T-t)\right)$$

Black-Scholes formula

$$\frac{C_K(t,T)}{Z(t,T)} = \mathbb{E}_*\left[\frac{(S_T-K)^+}{Z(T,T)}\middle| S_t\right]$$

$$= \mathbb{E}_*\left[\left(e^{\log S_T} - K\right)^+\middle| S_t\right]$$

$$= \int_{\log K}^{\infty} (e^y - K)\frac{1}{\sqrt{2\pi}\sigma\sqrt{T-t}} e^{-\frac{\left(y - \left[\log S_t + \left(r - \frac{1}{2}\sigma^2\right)(T-t)\right]\right)^2}{2\sigma^2(T-t)}} \mathrm{d}y$$



$$d_1 = \frac{\log\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

$$\frac{C_K(t,T)}{Z(t,T)} = \underbrace{\left(\frac{S_t}{Z(t,T)}\right)}_{F(t,T)}\Phi(d_1) - K\Phi(d_2)$$